LARGE CARDINALS, COMPACTNESS PROPERTIES AND RELATED RESULTS AROUND LOCALLY PROYECTIVE MODULES

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ABSTRACT. We introduce the class of κ-locally projective modules and prove that they have the compactness property for R when κ is a singular, a subtle (under $V = L$), or a weakly compact cardinal (for R a PID). In the case when κ singular, we show that Shelah's Singular Compactness Theorem holds for these modules. We also show that for some not weakly compact cardinal κ, κ-locally projective does not imply locally projective. Finally, we provide some results about ultraproducts of locally projective modules.

1. INTRODUCTION

The study of κ-free R-modules, that is, R-modules having the property that "most submodules" generated by $< \kappa$ elements are free, has been mainly focused on determining which pairs consisting of a cardinal κ and a ring R present the *compactness property*. By this, we mean to determine if every $\leq \kappa$-generated κ-free R-module is free (in which case we also say that κ *has*

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the compactness property for R). What "most submodules" means in this definition depends on the kind of ring R we are dealing with. In the case of κ-free abelian groups, "most submodules" simply means "all subgroups". However, for modules over arbitrary rings R, one cannot expect all submodules to be free, so "most submodules" will stand for a specific family of free submodules, with certain closure properties (see Chapter 4 of [9]).

In this paper, we use these ideas to introduce the class of κ-locally projective modules, which, by analogy, are those modules having the property that "most submodules" generated by $< \kappa$ elements are locally projective. The main results of this work show that κ-locally projective R-modules satisfy the compactness property for different kinds of cardinals κ and rings R. Also, we provide an example of a cardinal κ not having the compactness property for slender principal ideal domains. Before one can tackle compactness problems, one needs to be acquainted with known properties of the class of locally projective R-modules. These modules have received different names since they were introduced by *Gruson* and *Raynaud* in [13] as *flat strict Mittag-Leffler* modules: they were called *trace modules* by *Ohm* and *Rush* in[19], *universally torsionless modules* by *Garfinkel* in [10], and *locally projective modules* by *Zimmermann-Huisgen* in [21], which is the name we adhere to. These apparently different classes of modules are proven to be the same one (see [1] or [12]).

The paper is structured as follows. Section 2 is a survey of notions concerning locally projective modules that we consider relevant in order to approach compactness problems. For instance, characterizations of these modules when the ring R is a principal ideal domain (PID), and necessary and sufficient conditions for direct products of locally projective modules to be locally projective.

All the notions mentioned in this section are known but are scattered throughout the literature. In Section 3, we introduce the class of κ-locally projective modules and, by means of Shelah's Singular Compactness Theorem, we give a first result on compactness for λ-generated modules when λ is a singular cardinal larger than $|R|$. In Section 4, subtle cardinals and their basic properties are introduced. Section 5 presents the results related to the constructible universe L which will be necessary in Section 6, where we prove under $V = L$ that if κ is a subtle cardinal, then κ-locally projective modules of cardinality κ are locally projective. The proof of this result is very demanding and requires the construction in L of some elementary embeddings. In Section 7, it is shown that if κ is weakly compact, then κ-locally projective modules of cardinality κ over a PID are locally projective. We also prove that regular cardinals κ not weakly compact and less than any measurable cardinal do not have the compactness property for R by giving an example of a κ-locally projective module of cardinality κ which is not locally projective. Finally, in Section 8 we study the ultraproduct of locally projective modules when a measurable cardinal is the index set.

Throughout this work we will let $M = M_R$ be a right R-module, where R is an infinite associative ring with 1.

with $f_1^\alpha, \ldots, f_{k_\alpha}^\alpha \in M_\alpha^*$ and $x_1^\alpha, \ldots, x_{k_\alpha}^\alpha \in M_\alpha$. For $\ell < \omega$, put

$$U_\ell = \{\alpha < \kappa : k_\alpha = \ell\}$$

It follows that $\kappa = \dot{\bigcup}_{\ell<\omega} U_\ell$, so there is a (unique) $k < \omega$ such that $U_k \in \mathscr{U}$.

For $i = 1, \ldots, k$, $[n] \in \overline{M}$, $a \in [n]$ and $r \in R$, define

$$U_{i,a,r} = \{\alpha < \kappa : f_i^\alpha(a_\alpha) = r\}.$$

By a similar argument as above, for all $i = 1, \ldots, k$, there is an $r_{i,a} \in R$ for which $U_{i,a,r_{i,a}} \in \mathscr{U}$. Notice that if $b \in [n]$, then $r_{i,a} = r_{i,b}$. For all $i = 1, \ldots, k$, we define

$$f_i : \overline{M} \longrightarrow R$$

by means of $f_i([n]) = r_{i,n}$ for all $[n] \in \overline{M}$. Let $U = U_k \cap \bigcap_{i=1}^k U_{i,m,r_{i,m}} \in \mathscr{U}$. Then $[m'] = \sum_{i=1}^k [x_i] f_i([m])$, where $x_i = (x_i^\alpha : \alpha < \kappa)$. It follows that

$$U \subseteq \{\alpha < \kappa : m'_\alpha = m_\alpha\} \in \mathscr{U}$$

from which we get that $[m'] = [m]$ and $\overline{M}$ is locally projective. ❒

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